Bounded implies eventually periodic for the positive case of reciprocal-max difference equation with periodic parameters

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Abstract

This paper concerns the positive case of the difference equation

$$x_n = \max_{i \in \{1,2,\ldots,k\}} \left\{ \frac{(A_i)_n}{x_{n-i}} \right\}$$

with initial x terms positive, $(A_i)_n$ nonnegative, periodic in n and such that $\forall n\ \exists i$ such that $(A_i)_n > 0$. We show that if a solution is bounded then it is eventually periodic. Previous results exist for $k = 1$ and 2. We first make a log transformation, replacing division by subtraction, then define a dynamical system that is equivalent to the difference equation. This system is shown to be nonexpansive in L_∞. A theorem by Weller, [4], states that bounded solutions that are nonexpansive in a polyhedral norm, such as L_∞, have finite ω-limit sets. We prove that if a bounded solution has a finite ω-limit set then it must be eventually periodic. Therefore bounded implies eventually periodic for the log version. Finally, we apply this result to show that all positive solutions of the reciprocal difference equation with maximum are eventually periodic.

1 Introduction

This paper concerns the difference equation

$$x_n = \max_{i \in \{1,2,\ldots,k\}} \left\{ \frac{(A_i)_n}{x_{n-i}} \right\} \tag{1}$$

with initial x terms positive and $(A_i)_n$ nonnegative and periodic in n with the i^{th} component having period p_i. We also assume that for each n there exists $i \in \{1, 2, ..., k\}$ such that $(A_i)_n > 0$. This insures that each x_n is positive. The case $k = 2$ has been investigated. In [5] it was proven that Equation (1) with $k = 2$ is bounded if and only if neither p_1 nor p_2 are multiples of three.

We prove that for any size k all solutions that are bounded must be eventually periodic. This is accomplished by first transforming the system into a topologically conjugate system, which we call the log version. We show that the log version is nonexpansive in L_∞. A solution of Equation (1) must be both bounded and persistent for the log version to be bounded, but we show that for solutions of Equation (1) bounded and persistent are equivalent.

A theorem of Weller, [4], states that a bounded orbit of a discrete dynamical system that is nonexpansive in a polyhedral norm, such as L_∞, has a finite ω-limit set. This implies that a bounded orbit of Equation (1) has a finite ω-limit set. We show that solutions cannot approach their ω-limit sets arbitrarily closely, but must eventually jump onto the ω-limit set. Finally, we apply this result to show that all solutions with positive initial conditions of the reciprocal difference equation with maximum are eventually periodic.

2 Transformation into a nonexpansive dynamical system

Equation (1) takes an initial vector of k positive terms, $\vec{x} = (x_i)_{i=-k}^{-1}$, and generates a sequence starting with x_0. Since $\ln(x)$ is a diffeomorphism between $(0, \infty)$ and $\mathbb{R}$, we can make a transformation to a dynamically conjugate system, which we call the log version. Let

$$y_n = \ln(x_n), \quad \left(a_i\right)_n = \begin{cases} \ln\left(A_i\right)_n & \text{if } \left(A_i\right)_n > 0 \\ "-\infty" & \text{if } \left(A_i\right)_n = 0 \end{cases} \tag{2}$$

The log version is

$$y_n = \max_{i \in \{1,2,\ldots,k\}} \left\{\left(a_i\right)_n - y_{n-i}\right\} \tag{3}$$

where $\forall i \in \{1, 2, ..., k\}$, $\left(a_i\right)_n$ is periodic with period p_i. Note both y_n and $\left(a_i\right)_n$ may range over all real numbers, including zero. Let J be the smallest number larger than or equal to k that is a common multiple of the periods $\{p_1, p_2, ..., p_k\}$. Use Equation (3) to define the dynamical system

$$f : \mathbb{R}^J \to \mathbb{R}^J \text{ by } f\left((y_i)_{i=-J}^{-1}\right) = (y_i)_{i=0}^{J-1}. \tag{4}$$

Since the dimension of the domain and range of f is a common multiple of the periods of all the parameters, the orbits of the autonomous dynamical system f correspond to the orbits of Equation (3). If $J > k$, define $(y_i)_{i=-J}^{-k-1} = \vec{0}$ to obtain a fixed initial condition for Equation (3). Although the domain of f is J dimensional, only the last k elements affect the output. Note that $f\left((y_i)_{i=n}^{n+J-1}\right)$ is assured to correspond to $(y_i)_{i=n+J}^{n+2J-1}$ if $n \bmod J = 0$.

Solutions of Equation (1) must be both bounded and persistent for the log version solution to be bounded, but the following lemma shows that bounded and persistent are equivalent for Equation (1).

Lemma 1 *A solution of Equation (1) persists if and only if it is bounded.*

Proof. Suppose a solution of Equation (1) is bounded. Let M be an upper bound on the x terms. The set of positive, distinct values of $\left(A_i\right)_n$ is finite, so it must have a minimum. Let c be that minimum. Then the smallest possible x value is $\frac{c}{M}$, so the solution persists.
Similarly, suppose a solution of Equation (1) persists. Let $L > 0$ be a lower bound on the x values, and let d be the maximum of the $\left(A_i\right)_n$ values. Then $\frac{d}{L}$ is an upper bound on the x terms. ■

2.1 An example

The system

$$x_n = max\left\{\frac{A_n}{x_{n-1}}, \frac{B_n}{x_{n-2}}\right\}$$

$$A_n = \{8, 2, 8, 2, ...\} \text{ period } 2, \quad B_n = \{7, 6, 5, 4, 3, 7, 6, 5, 4, 3, ...\} \text{ period } 5$$

has $k = 2$, and neither period is a factor of 3, so it is bounded. $J = \mathrm{lcm}(2, 5) = 10 > k = 2$, and the dynamical system $f : \mathbb{R}^{10} \to \mathbb{R}^{10}$ is

$$f\left((y_i)_{i=0}^{9}\right) = (y_i)_{i=10}^{19} = \begin{cases} y_{10} = \max\{\ln(8) - y_9, \ln(7) - y_8\} \\ y_{11} = \max\{\ln(2) - y_{10}, \ln(6) - y_9\} \\ y_{12} = \max\{\ln(8) - y_{11}, \ln(5) - y_{10}\} \\ y_{13} = \max\{\ln(2) - y_{12}, \ln(4) - y_{11}\} \\ y_{14} = \max\{\ln(8) - y_{13}, \ln(3) - y_{12}\} \\ y_{15} = \max\{\ln(2) - y_{14}, \ln(7) - y_{13}\} \\ y_{16} = \max\{\ln(8) - y_{15}, \ln(6) - y_{14}\} \\ y_{17} = \max\{\ln(2) - y_{16}, \ln(5) - y_{15}\} \\ y_{18} = \max\{\ln(8) - y_{17}, \ln(4) - y_{16}\} \\ y_{19} = \max\{\ln(2) - y_{18}, \ln(3) - y_{17}\} \end{cases}$$

The initial condition $(0, 0, 0, 0, 0, 0, 0, 0, 2, 1)$ is eventually periodic of period 2. The first image is

$$\begin{gathered}(3ln(2) - 1, ln(6) - 1, 3ln(2) - ln(6) + 1, 2ln(2) - ln(6) + 1, ln(2) + ln(6) - 1, \\ ln(7) - 2ln(2) + ln(6) - 1, 5ln(2) - ln(7) - ln(6) + 1, ln(5) - ln(7) + 2ln(2) - ln(6) + 1, \\ ln(2) - ln(5) + ln(7) + ln(6) - 1, ln(3) - ln(5) + ln(7) - 2ln(2) + ln(6) - 1).\end{gathered}$$

This and the second image are not periodic but the third image is periodic of period 2. We will show that initial conditions becoming periodic is the general situation for the reciprocal-max difference equation.

2.2 The dynamical system is nonexpansive

We show that f is nonexpansive in L_∞. This means that for $y, z \in \mathbb{R}^J$, $\|f(z) - f(y)\|_\infty \leq \|z - y\|_\infty$.

Theorem 2 *System (4) is nonexpansive in L_∞.*

Proof. Consider two initial vectors $\vec{y}$, and $\vec{z}$, which generate solutions by Equation (3). Let $n \in \mathbb{N}$ be fixed. Let $c = \max\{|y_i - z_i| \mid i \in \{n-k, n-k+1, ..., n-1\}\}$. We prove that $|y_n - z_n| \leq c$, which by induction proves that System (4) is nonexpansive.
Without loss of generality assume that $y_n > z_n$. Let $j \in \{1, 2, ..., k\}$ such that $y_n = \left(a_j\right)_n - y_{n-j}$. Then $z_n \geq \left(a_j\right)_n - z_{n-j}$. Therefore
$0 \leq y_n - z_n = \left(a_j\right)_n - y_{n-j} - z_n \leq \left(a_j\right)_n - y_{n-j} - \left(\left(a_j\right)_n - z_{n-j}\right) = z_{n-j} - y_{n-j} \leq c$. ■

3 Bounded implies eventually periodic.

System (4) is a nonexpansive dynamical system in a polyhedral norm, so if an initial vector has a bounded solution then it's ω-limit set is finite, as was proven by Weller [4].

The next lemma shows that if y, an initial vector, has a finite ω-limit set under f then this set consists of a single periodic orbit. It also shows that if P is the period of this orbit then $\lim_{n\to\infty} f^{nP}(y)$ exists.

Lemma 3 *Let $f : \mathbb{R}^J \to \mathbb{R}^J$ and $y \in \mathbb{R}^J$. If $\omega(y)$ has cardinality $P \in \mathbb{N}$, then $\omega(y)$ consists of a single periodic orbit of period P and $\lim_{n\to\infty} f^{nP}(y)$ exists.*

Proof. Let $f : \mathbb{R}^J \to \mathbb{R}^J$ with metric d and let $y \in \mathbb{R}^J$ with $\omega(y)$ having cardinality $P \in \mathbb{N}$. Let $\varepsilon > 0$ be less than one-half of the minimum distance between the points in $\omega(y)$. By the continuity of f there is a positive δ smaller than ε such that if the distance from $x \in \mathbb{R}^J$ to some point $z \in \omega(y)$ is less than δ, then $d(f(x), f(z)) < \varepsilon$. The orbit of y has a subsequence $(f^{n_k}(y))$ which converges to z. Hence there exits $K \in \mathbb{N}$ such that $k \geq K$ implies that $d(f^{n_k}(y), z) < \delta$. Thus $d(f^{n_k+1}(y), f(z)) < \varepsilon$. If in fact $d(f^{n_k+1}(y), f(z)) < \delta$, then $d(f^{n_k+2}(y), f^2(z)) < \varepsilon$ since the orbit of z is a subset of $\omega(y)$. Iteration of this shows that either there is a k^* such that $d(f^{n_{k^*}+s}(y), f^s(z)) < \delta$ for all $s \in \mathbb{N}$ or for each k there is an s_k with $\delta \leq d(f^{n_k+s_k}(y), f^{s_k}(z)) < \varepsilon$. In the later case the sequence $(f^{n_k+s_k}(y))$ is always more than δ from any point in $\omega(y)$. This is impossible since this sequence must have a convergent subsequence which would be an element of $\omega(y)$. So there is a k^* such that $d(f^{n_{k^*}+s}(y), f^s(z)) < \delta$ for all $s \in \mathbb{N}$. The orbit of $f^{n_{k^*}}(y)$ must get within δ of every point in $\omega(y)$, but it also stays within δ of the orbit of z which is a subset of $\omega(y)$. Hence the orbit of z must equal $\omega(y)$. Hence z must be periodic and its orbit is $\omega(y)$.

To finish the proof note that since $d(f^{n_{k^*}+sP}(y), f^{sP}(z)) < \delta$ for all $s \in \mathbb{N}$ and $f^{sP}(z) = z$, the bounded sequence $(f^{n_{k^*}+sP}(y))$ has subsequences that can only converge to z. Thus this sequence must converge to z and $\lim_{n\to\infty} f^{nP}(y)$ is one of the points in the orbit of z. ■

The following theorem shows that if some initial condition for f has a finite ω-limit set then its orbit must be eventually periodic.

Theorem 4 *Let $(y_i)_{i=-J}^{-1}$ be an initial condition that has a bounded orbit under System (4). Then the sequence $(y_i)_{i=0}^{\infty}$ is eventually periodic.*

Proof. Let $y = (y_i)_{i=-J}^{-1}$ be an initial condition for System (4) such that the sequence $(y_i)_{i=0}^{\infty}$ determined by the underlying Equation (3) is bounded. Let P be the cardinality of $\omega(y)$, the ω-limit set of y. Weller [4] showed that P is finite since System (4) is a nonexpansive dynamical system in a polyhedral norm. Let $\dot{y} = (\dot{y}_i)_{i=0}^{J-1} = \lim_{n\to\infty} f^{nP}(y)$. Lemma 3 shows that $\dot{y}$ exists and is a periodic point of period P in $\omega(y)$. Since $\dot{y}$ is periodic of period P, its orbit can be viewed as generating the periodic sequence $(\dot{y}_i)_{i=0}^{PJ-1}$. Now define the periodic sequence of values $(d_j)_{j=0}^{JP-1}$ by

$$d_j = \min\left(\{1\} \cup \left\{\dot{y}_j - \left(a_i\right)_j + \dot{y}_{j-i} \mid \dot{y}_j > \left(a_i\right)_j - \dot{y}_{j-i}, i \in \{1,2,...,k\}\right\}\right).$$

Since $\dot{y}_j = \max\limits_{i\in\{1,2,...,k\}} \left(a_i\right)_j + \dot{y}_{j-i}$, d_j is the smallest positive difference between $\dot{y}_j$ and the terms in its log-max equation, or 1. The minimum with 1 is in case the set of positive differences is empty. Thus $d_j > 0$ for all $j \in \{0,1,\cdots,JP-1\}$. Let $\epsilon = \frac{1}{3}\min\{d_j \mid j \in \{0,1,\cdots,JP-1\}\}$. Note $\epsilon > 0$. Let $U_\epsilon = \left\{z \in \mathbb{R}^J \mid \|z - \dot{y}\|_\infty < \epsilon\right\}$. Since $\lim_{n\to\infty} f^{nP}(y) = \dot{y}$, there is an $n^* \in \mathbb{N}$ such that $f^{n^*P}(y) \in U_\epsilon$. Let $f^{n^*P}(y) = z = (z_i)_{i=0}^{J-1}$, then $z_i = y_{i-J+n^*PJ}$ for all $i \geq 0$. Theorem 2 shows that $|z_i - \dot{y}_i| < \epsilon$ for all $i \geq 0$. For $i \in \{0,1,\cdots,J-1\}$ let $\Delta z_i = |\dot{y}_i - z_i|$ and let $D = \{0\} \cup \{\Delta z_i \mid i \in \{0,1,\cdots,J-1\}\}$.

Claim 5 *Let $j \in \{1,2,\cdots,k\}$. If $z_n = \left(a_j\right)_n - z_{n-j}$ then $\dot{y}_n = \left(a_j\right)_n - \dot{y}_{n-j}$.*

Proof of Claim. This claim shows that the term selected in the log-max equation of z_n also works for $\dot{y}_n$. Let $j \in \{1,2,\cdots,k\}$ *s.t.* $z_n = \left(a_j\right)_n - z_{n-j}$. Suppose $\dot{y}_n \neq \left(a_j\right)_n - \dot{y}_{n-j}$. Then

$$\begin{gathered}
0 < \epsilon \leq \frac{d_n}{3} \leq \frac{\dot{y}_n - (a_j)_n + \dot{y}_{n-j}}{3}, \\
3\epsilon \leq \dot{y}_n - \left(a_j\right)_n + \dot{y}_{n-j}, \\
\dot{y}_{n-j} \geq 3\epsilon + \left(a_j\right)_n - \dot{y}_n, \\
\dot{y}_{n-j} - z_{n-j} \geq 3\epsilon + \left(a_j\right)_n - \dot{y}_n - (\left(a_j\right)_n - z_n) \text{ and} \\
\dot{y}_{n-j} - z_{n-j} \geq 3\epsilon + (z_n - \dot{y}_n)
\end{gathered}$$

which is impossible since $\dot{y}_{n-j} - z_{n-j} < \epsilon$ and $z_n - \dot{y}_n > -\epsilon$. Therefore it must be true that $\dot{y}_n = \left(a_j\right)_n - \dot{y}_{n-j}$.

This claim shows that there is a $j \in \{1,2,\cdots,k\}$ such that $|z_J - \dot{y}_J| = \left|\left(a_j\right)_J - z_{J-j} - (\left(a_j\right)_J - \dot{y}_{J-j})\right| = |z_{J-j} - \dot{y}_{J-j}| = \Delta z_{J-j} \in D$. By repeating this argument $|z_i - \dot{y}_i| \in D \;\forall\, i \geq J$. Now let $\delta = \min\ \{\Delta z_i \mid \Delta z_i > 0, i \in \{0,1,\cdots,J-1\}\}$. Let $m \in \mathbb{N}$ *s.t.* $\|f^{mP}(z) - \dot{y}\|_\infty < \delta$. Then
$\max\{|z_{JmP+i} - \dot{y}_i| \mid i \in \{0,1,\cdots,J-1\}\} < \delta$, and

$\max\{|z_{JmP+i} - \dot{y}_i| \mid i \in \{0, 1, \cdots, J-1\}\} \in D$, so
$\max\{|z_{JmP+i} - \dot{y}_i| \mid i \in \{0, 1, \cdots, J-1\}\} = 0$, and
$\dot{y} = f^{mP}(z) = f^{(n*+m)P}(y)$, and therefore $f^{(n*+m+1)P}(y) = f^{(n*+m)P}(y)$, so $(y_i)_{i=0}^{\infty}$ is an eventually periodic sequence with period JP. ■

4 The positive case of the reciprocal difference equation with maximum is eventually periodic.

Theorem (4) showed that bounded implies eventually periodic for Equation (1). An important corollary to this is that the positive case of the reciprocal difference equation with maximum is eventually periodic. This difference equation is

$$x_n = \max_{i \in \{1,2,\ldots,k\}} \left\{ \frac{B_i}{x_{n-i}} \right\}, \quad B_i \text{ nonnegative, } \vec{x} \text{ positive.} \tag{5}$$

Voulov has shown in [6] that Equation (5) is bounded, and since Equation (5) is just a special case of Equation (1), its solutions are eventually periodic.

Corollary 6 *All solutions of Equation (5) are eventually periodic.*

Proof. Let $k \in \mathbb{N}$, and $B_i \geq 0$ for $i \in \{1, 2, \cdots, k\}$ be fixed. Use these to define Equation (5). Define $(A_i)_n$ by

$$(A_i)_n = B_i \; \forall \, n \in \mathbb{N}, \; i \in \{1, 2, \cdots, k\}. \tag{6}$$

Then these $(A_i)_n$ values define an equivalent difference Equation (1). Since Equation (5) is bounded, the equivalent difference Equation (1) is also bounded. By Theorem (4) the solutions of Equation (1) are eventually periodic, so the solutions of Equation (5) are also eventually periodic. ■

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