

### Persistence of Fixed Point Theorem

Let  $f_\lambda(x)$  be  $C^1$  and  $f_{\lambda_0}(x_0) = x_0$ . Assume that  $f'_{\lambda_0}(x_0) \neq 1$ . Then there is a rectangle  $U$  about  $(\lambda_0, x_0)$ , an interval  $I_\delta(\lambda_0)$  and

$$g : I_\delta(\lambda_0) \rightarrow \mathbb{R} \text{ such that } g(\lambda_0) = x_0, \quad g'(\lambda_0) = \frac{\frac{\partial f_\lambda(x_0)}{\partial \lambda} \Big|_{\lambda=\lambda_0}}{f'_{\lambda_0}(x_0) - 1}$$

and

$$f(\lambda, x) \in U : f_\lambda(x) = x \iff f(\lambda, x) \in U : x = g(\lambda), \lambda \in I_\delta(\lambda_0) \iff g.$$

### Tangent Bifurcation Theorem (Saddle-Node bifurcation)

Let  $f_\lambda(x)$  be  $C^2$  and  $f_{\lambda_0}(x_0) = x_0$ . Assume that

- $f'_{\lambda_0}(x_0) = 1$ .
- $f''_{\lambda_0}(x_0) \neq 0$ .
- $\frac{\partial f_\lambda(x_0)}{\partial \lambda} \Big|_{\lambda=\lambda_0} \neq 0$ .

Then there is a rectangle  $U$  about  $(\lambda_0, x_0)$ , an interval  $I_\delta(x_0)$  and

$$p : I_\delta(x_0) \rightarrow \mathbb{R} \text{ such that } p(x_0) = \lambda_0, \quad p'(x_0) = 0, \quad p''(x_0) = \frac{f''_{\lambda_0}(x_0)}{\frac{\partial f_\lambda(x_0)}{\partial \lambda} \Big|_{\lambda=\lambda_0}} \neq 0$$

and

$$f(\lambda, x) \in U : f_\lambda(x) = x \iff f(\lambda, x) \in U : \lambda = p(x) \iff g.$$

### Simple Pitchfork Bifurcation

Let  $f_\lambda(x)$  be  $C^3$  and  $f_{\lambda_0}(0) = 0$ . Assume that

- $f_\lambda(0) = 0$  for all  $\lambda$  near  $\lambda_0$ .
- $f''_{\lambda_0}(0) = 1$ .
- $f'''_{\lambda_0}(0) = 0$ .
- $f'''_{\lambda_0}(0) \neq 0$ .
- $\frac{\partial f'_\lambda(0)}{\partial \lambda} \Big|_{\lambda=\lambda_0} \neq 0$ .

Then there is a rectangle  $U$  about  $(\lambda_0, 0)$ , an interval  $I_\delta(0)$  and

$$p(x) : I_\delta(0) \rightarrow \mathbb{R} \text{ such that } p(0) = \lambda_0, p'(0) = 0, p''(0) = i \frac{f'''_{\lambda_0}(0)}{3 \frac{\partial f'_\lambda(0)}{\partial \lambda} \Big|_{\lambda=\lambda_0}} \neq 0$$

and

$$f(\lambda, x) \in U : f_\lambda(x) = x \text{ and } x \neq 0 \iff f(\lambda, x) \in U : \lambda = p(x), \text{ and } x \neq 0.$$

### Simple Period Doubling Bifurcation

Let  $f_\lambda(x)$  be  $C^3$  and  $f_{\lambda_0}(0) = 0$ . Assume that

- $f_\lambda(0) = 0$  for all  $\lambda$  near  $\lambda_0$ .
- $f''_{\lambda_0}(0) = -1$ .
- $f'''_{\lambda_0}(0) \neq 0$ .
- $\frac{\partial (f''_\lambda(0))}{\partial \lambda} \Big|_{\lambda=\lambda_0} \neq 0$ .

Then there is a rectangle  $U$  about  $(\lambda_0, 0)$ , an interval  $I_\delta(0)$  and

$$p(x) : I_\delta(0) \rightarrow \mathbb{R} \text{ such that } p(0) = \lambda_0, p'(0) = 0, p''(0) = i \frac{f'''_{\lambda_0}(0)}{3 \frac{\partial (f''_\lambda(0))}{\partial \lambda} \Big|_{\lambda=\lambda_0}} \neq 0$$

and

$$\textcircled{c} (\lambda, x) \in U : f^2_\lambda(x) = x \text{ and } x \neq 0 \iff \textcircled{a} f(\lambda, x) \in U : \lambda = p(x), \text{ and } x \neq 0.$$

### Period Doubling Bifurcation

Let  $f_\lambda(x)$  be  $C^3$  and  $f_{\lambda_0}(x_0) = x_0$ . Assume that

- $f'_{\lambda_0}(x_0) = -1$ .
- $f''_{\lambda_0}(x_0) \neq 0$ .
- $\frac{\partial (f'_\lambda)^0(x_0)}{\partial \lambda} \Big|_{\lambda=\lambda_0} \neq 0$ .

Then there is a rectangle  $U$  about  $(\lambda_0, x_0)$ , an interval  $I_\delta(x_0)$  and

$$p(x) : I_\delta(x_0) \rightarrow \mathbb{R} \text{ such that } p(x_0) = \lambda_0, p'(x_0) = 0, p''(x_0) = \frac{f''_{\lambda_0}(x_0)}{3 \frac{\partial (f'_\lambda)^0(x_0)}{\partial \lambda} \Big|_{\lambda=\lambda_0}} \neq 0$$

and

$$\textcircled{c} \quad \{(\lambda, x) \in U : f_\lambda^2(x) = x \text{ and } f_\lambda(x) \neq x\} = \{(\lambda, x) \in U : \lambda = p(x), \text{ and } x \neq x_0\}.$$

### Simple Transcritical Bifurcation Theorem

Let  $f_\lambda(x)$  be  $C^2$  and  $f_{\lambda_0}(0) = 0$ . Assume that

- $f_\lambda(0) = 0$  for all  $\lambda$ .
- $f'_{\lambda_0}(0) = 1$ .
- $f''_{\lambda_0}(0) \neq 0$ .
- $\frac{\partial f'_\lambda(0)}{\partial \lambda} \Big|_{\lambda=\lambda_0} \neq 0$ .

Then there is a rectangle  $U$  about  $(\lambda_0, 0)$ , an interval  $I_\delta(0)$  and

$$p(x) : I_\delta(0) \rightarrow \mathbb{R} \text{ such that } p'(0) \neq 0$$

and

$$\{(\lambda, x) \in U : f_\lambda(x) = x \text{ and } x \neq 0\} = \{(\lambda, x) \in U : \lambda = p(x), x \in I_\delta(0) \text{ and } x \neq 0\}.$$