Persistence of Fixed Point Theorem

Let $f_\lambda(x)$ be C^1 and $f_{\lambda_0}(x_0) = x_0$. Assume that $f\,'_{\lambda_0}(x_0) \neq 1$. Then there is a rectangle U about (λ_0, x_0), an interval $I_\delta(\lambda_0)$ and

$$g(\lambda) : I_\delta(\lambda_0) \to \mathbb{R} \text{ such that } g(\lambda_0) = x_0,\ g'(\lambda_0) = -\frac{\left.\frac{\partial f_\lambda(x_0)}{\partial \lambda}\right|_{\lambda=\lambda_0}}{f\,'_{\lambda_0}(x_0) - 1}$$

and

$$\{(\lambda, x) \in U : f_\lambda(x) = x\} = \{(\lambda, x) \in U : x = g(\lambda), \lambda \in I_\delta(\lambda_0)\}.$$

Tangent Bifurcation Theorem (Saddle-Node bifurcation)

Let $f_\lambda(x)$ be C^2 and $f_{\lambda_0}(x_0) = x_0$. Assume that

a. $f\,'_{\lambda_0}(x_0) = 1$.

b. $f\,''_{\lambda_0}(x_0) \neq 0$.

c. $\left.\frac{\partial f_\lambda(x_0)}{\partial \lambda}\right|_{\lambda=\lambda_0} \neq 0$.

Then there is a rectangle U about (λ_0, x_0), an interval $I_\delta(x_0)$ and

$$p(x) : I_\delta(x_0) \to \mathbb{R} \text{ such that } p(x_0) = \lambda_0, p'(x_0) = 0, p''(x_0) = -\frac{f\,''_{\lambda_0}(x_0)}{\left.\frac{\partial f_\lambda(x_0)}{\partial \lambda}\right|_{\lambda=\lambda_0}} \neq 0$$

and

$$\{(\lambda, x) \in U : f_\lambda(x) = x\} = \{(\lambda, x) \in U : \lambda = p(x)\}.$$

Simple Pitchfork Bifurcation

Let $f_\lambda(x)$ be C^3 and $f_{\lambda_0}(0) = 0$. Assume that

a. $f_\lambda(0) = 0$ for all λ near λ_0.

b. $f\,'_{\lambda_0}(0) = 1$.

c. $f\,''_{\lambda_0}(0) = 0$.

d. $f\,'''_{\lambda_0}(0) \neq 0$.

e. $\left.\frac{\partial f\,'_\lambda(0)}{\partial\lambda}\right|_{\lambda=\lambda_0} \neq 0$.

Then there is a rectangle U about $(\lambda_0, 0)$, an interval $I_\delta(0)$ and

$$p(x) : I_\delta(0) \to \mathbb{R} \text{ such that } p(0) = \lambda_0, p'(0) = 0, p''(0) = -\frac{f\,'''_{\lambda_0}(0)}{3\left.\frac{\partial f\,'_\lambda(0)}{\partial\lambda}\right|_{\lambda=\lambda_0}} \neq 0$$

and

$$\{(\lambda, x) \in U : f_\lambda(x) = x \text{ and } x \neq 0\} = \{(\lambda, x) \in U : \lambda = p(x), \text{ and } x \neq 0\}.$$

Simple Period Doubling Bifurcation

Let $f_\lambda(x)$ be C^3 and $f_{\lambda_0}(0) = 0$. Assume that

a. $f_\lambda(0) = 0$ for all λ near λ_0.

b. $f\,'_{\lambda_0}(0) = -1$.

c. $\left(f^2_{\lambda_0}\right)'''(0) \neq 0$.

d. $\left.\frac{\partial (f^2_\lambda)'(0)}{\partial\lambda}\right|_{\lambda=\lambda_0} \neq 0$.

Then there is a rectangle U about $(\lambda_0, 0)$, an interval $I_\delta(0)$ and

$$p(x) : I_\delta(0) \to \mathbb{R} \text{ such that } p(0) = \lambda_0, p'(0) = 0, p''(0) = -\frac{\left(f^2_{\lambda_0}\right)'''(0)}{3\left.\frac{\partial (f^2_\lambda)'(0)}{\partial\lambda}\right|_{\lambda=\lambda_0}} \neq 0$$

and

$$\{(\lambda, x) \in U : f^2_\lambda(x) = x \text{ and } x \neq 0\} = \{(\lambda, x) \in U : \lambda = p(x), \text{ and } x \neq 0\}.$$

Period Doubling Bifurcation

Let $f_\lambda(x)$ be C^3 and $f_{\lambda_0}(x_0) = x_0$. Assume that

a. $f\,'_{\lambda_0}(x_0) = -1$.

b. $\left(f^2_{\lambda_0}\right)'''(x_0) \neq 0$.

c. $\left.\frac{\partial (f^2_\lambda)'(x_0)}{\partial \lambda}\right|_{\lambda=\lambda_0} \neq 0$.

Then there is a rectangle U about (λ_0, x_0), an interval $I_\delta(x_0)$ and

$$p(x) : I_\delta(x_0) \to \mathbb{R} \text{ such that } p(x_0) = \lambda_0, p'(x_0) = 0, p''(x_0) = -\frac{\left(f^2_{\lambda_0}\right)'''(x_0)}{3\left.\frac{\partial (f^2_\lambda)'(x_0)}{\partial \lambda}\right|_{\lambda=\lambda_0}} \neq 0$$

and

$$\left\{(\lambda, x) \in U : f^2_\lambda(x) = x \text{ and } f_\lambda(x) \neq x\right\} = \{(\lambda, x) \in U : \lambda = p(x), \text{ and } x \neq x_0\}.$$

Simple Transcritical Bifurcation Theorem

Let $f_\lambda(x)$ be C^2 and $f_{\lambda_0}(0) = 0$. Assume that

a. $f_\lambda(0) = 0$ for all λ.

b. $f\,'_{\lambda_0}(0) = 1$.

c. $f\,''_{\lambda_0}(0) \neq 0$.

d. $\left.\frac{\partial f\,'_\lambda(0)}{\partial \lambda}\right|_{\lambda=\lambda_0} \neq 0$.

Then there is a rectangle U about $(\lambda_0, 0)$, an interval $I_\delta(0)$ and

$$p(x) : I_\delta(0) \to \mathbb{R} \text{ such that } p'(0) \neq 0$$

and

$$\{(\lambda, x) \in U : f_\lambda(x) = x \text{ and } x \neq 0\} = \{(\lambda, x) \in U : \lambda = p(x), x \in I_\delta(0) \text{ and } x \neq 0\}.$$